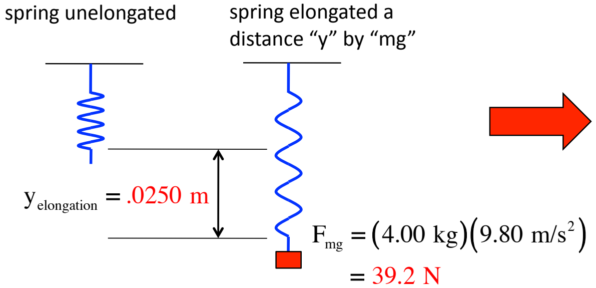
Problem 7.17

Heehee . . . fun stuff! You are trying to figure out how much work a spring does on a hanging mass that elongates the spring.

One of the parameters you inevitably need in problems like this is the *spring constant* "k." This constant tells you how much force is required to elongate the spring *per unit meter*. The easiest way to experimentally determine that constant is to attach a known force to the spring (this could be a known mass whereupon the force would be "mg"), see how far the force elongates the spring "y", then take the ratio F/y. In fact, that's exactly the information given in the preamble of the problem. Pictorially presented below:



$y_{elongation} = .0250 \text{ m}$

$F_{mg} = (4.00 \text{ kg})(9.80 \text{ m/s}^2) = 39.2 \text{ N}$

$$k = \frac{F_{mg}}{y_{elongate}} = \frac{(39.2 \text{ N})}{(.0250 \text{ m})} = 1570 \text{ N/m}$$

With the spring constant, we are ready to go to work (pun intended . . .)

a.) What is the elongation if a 1.50 kg block is attached to the spring?

According to Hooke's Law, the force on a spring is proportional to the displacement of the spring, with the proportionality constant being *minus* the *spring constant*. That is:

$$F = -ky$$

Noting that the force is just the weight of the block (mg), we can write:

$$\begin{aligned} F &= -ky \\ \Rightarrow \quad |y| &= \frac{F}{k} \\ &= \frac{mg}{k} \\ &= \frac{(1.50 \text{ kg})(9.80 \text{ m/s}^2)}{(1570 \text{ N/m})} \\ &= 9.36\text{x}10^{-3} \text{ m} \end{aligned}$$

b.) So now comes the fun. How much work does the spring do when elongated by a 1.50 kg hanging mass?

We know that the force gets larger as the elongation gets bigger, so we can't use the simplified $W_F = \vec{F} \bullet \vec{d}$ expression, we have to use the $W_F = \int \vec{F} \bullet d\vec{r}$ expression. Doing so yields:

$$\begin{aligned} W &= \int dW = \int \vec{F} \bullet d\vec{r} \\ \Rightarrow \quad W &= \int |\vec{F}||d\vec{r}| \cos\theta \\ &= \int_{y=0}^{4.00\text{x}10^{-2}} (ky)(dy)\cos 0^\circ \end{aligned}$$

Before we solve this, you might wonder why the angle is zero degrees. Answer: The question asked for the work done by "an external agent," which is to say, "you." You will be pulling on the spring in the same direction as the spring is elongating, so the angle between the force and displacement in the dot product will be zero degrees. This may seem like a triviality, but it doesn't hurt to take a moment to think about what the math is doing. It is really easy to get yourself balled up in signs if and when you ever try one of these problems completely on your own without the aid of the solutions.

Finishing off the problem:

$$\begin{aligned} W &= \int dW = \int \vec{F} \bullet d\vec{r} \\ \Rightarrow \quad W &= \int |\vec{F}||d\vec{r}| \cos\theta \\ &= \int_{y=0}^{4.00\text{x}10^{-2}} (|-ky|)(dy)\cancelto{1}{\cos 0^\circ} \\ &= \int_{y=0}^{4.00\text{x}10^{-2}} (ky)(dy) \\ &= k\left(\frac{y^2}{2}\right)\Bigg|_{y=0}^{4.00\text{x}10^{-2}} \\ &= \frac{1}{2}(1570 \text{ N/m})\left[(4.00\text{x}10^{-2} \text{ m})^2 - (0)^2\right] \\ &= 1.26 \text{ J} \end{aligned}$$

EXTRA (don't read this is you are in a hurry): As a preamble to what you are about the learn in class, using the Work/Energy theorem coupled with the fact that the initial and final spring velocities are both zero and the work the spring does is *negative* (I'm unembedding that negative in the relationship below), we can write:

$$
\begin{aligned}
&W_{net} = \Delta KE \\
\Rightarrow\quad & W_{you} + \left(-W_{spring}\right) = \cancelto{0}{\Delta KE} \\
\Rightarrow\quad & W_{you} = W_{spring}
\end{aligned}
$$

In short, if you can determine the work the spring does, you know the work *you* had to do. Before, you determined the work you did by grunting your way through that integral. As much fun at that was, there is an easier way.

Specifically, if the force field due to the spring is "conservative" (this will be defined for you later), and it is, you can determine how much work that field does as a body moves through it by determine the force field's "potential energy function *U*," then by executing the operation:

$$W_{cons.force} = -\Delta U$$

The potential energy function for a spring, which will be derived for you later, is:

$$U_{spring} = \frac{1}{2}kx^2$$

Using that *potential energy function* and the fact that the spring was initially unstretched, you can determine *the work the spring did* as it elongated as:

$$
\begin{aligned}
W_{spring} &= -\left(U_2 - U_1\right) \\
&= -\left(\frac{1}{2}kx_2^2 - \cancelto{0}{\frac{1}{2}kx_1^2}\right) \\
&= -\frac{1}{2}\left(1570 \text{ N/m}\right)\left(4.00\text{x}10^{-2} \text{ m}\right)^2 \\
&= -1.26 \text{ J}
\end{aligned}
$$

"Minus" this is the work *you* did (with no integral involved).

Bottom line: You will need to use the Calculus to derive the potential energy function for a spring, but once you've done that, you can easily do work calculations associated with springs by simply using their potential energy function and the fact that $W_{spring} = -\Delta U_{spring}$.